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Reminder

A group G is a set with binary operation s.t.

0) $\forall\ g, h \in G, g\cdot h \in G$ (closure)

1) $g(hk) = (gh)k\ \forall\ g, h, k \in G$ (associativity)

2) $\exists\ e \in G$ s.t. $g\cdot e = e\cdot g = g\ \forall\ g \in G$ (identity)

3) $\forall\ g \in G\ \exists\ h \in G$ s.t. $h\cdot g = g\cdot h = e$ (inverse)

Groups with maps (homomorphisms)

A subgroup $H\ G$ is normal if $\forall\ g \in G$

$$gHg^{-1} = H$$
$$ghg^{-1} = h_1 \in H$$

Homomorphisms from G ↔ normal subgroups of G [First Isomorphism Theorem for Groups]

§1 Factor groups and isomorphism theorems

G a group. $N \lhd G$ [[**nb** $\lhd$ denotes normal subgroup $\leq$ denotes any subgroup]]

Factor set $\frac{G}{N} = \{eN, g_1N, g_2N, \ldots\} = \{Ne, Ng_1, Ng_2, \ldots\}$

Theorem 1.1

$\frac{G}{N}$ with operation $(Ng_1)(Ng_2) = Ng_1g_2$ is a group

Proof (correctness)

If $Ng_1 = Ng_1'$ and $Ng_2 = Ng_2'$ then $(Ng_1')(Ng_2') =$

$Ng_1'g_2' = Nn_1g_1n_2g_3 = Ng_1n_2g_2 = Nn_3g_1g_2 = Ng_1g_2$ $[n_1, n_2, n_3 \in N, g_1n_2 = n_3g_1]$

(axioms)

Closure - clear by def.

Assoc $((Ng_1)(Ng_2))Ng_3 = (Ng_1g_2)Ng_3 = N(g_1g_2)g_3$

$(Ng_1)((Ng_2)(Ng_3)) = (Ng_1)(Ng_2g_3) = Ng_1(g_2g_3)$

Identity $N\cdot e$

Inverse $(Ng)^{-1} = N\cdot g^{-1}$

Definition

A map $\theta : G \to \frac{G}{N}; g \to Ng$ is called canonical homomorphism.

Theorem 1.2

θ is a group homomorphism, its image is $\frac{G}{N}$ and $Ker(\theta) = N$

Proof

1) $\theta(g_1g_2) = Ng_1g_2 = Ng_1Ng_2 = \theta(g_1)\theta(g_2)\ \forall\ g_1, g_2 \in G$

So θ is a homomorphism

2) Every coset in $\frac{G}{N}$ has form Ng for some g, so $\Im = \frac{G}{N}$

3) $Ker(\theta) = \{g \in G : gN = N\} = \{g \in G : g = n_1^{-1} n_2 \in N\} = N$

First Isomorphism Theorem

Let $\theta : G \to H$ be a homomorphism.
Then

(a) $\Im(\theta)\ H$
(b) $Ker(\theta) \trianglelefteq G$
(c) $\dfrac{G}{Ker(\theta)} \cong \Im(\theta)$

Proof

a) Let $h_i = \theta(g_i)$ for some $g_i \in G$

$h_1 \cdot h_2 = \theta(g_i) \cdot_H \theta(g_2) = \theta(g_1 \cdot_G g_2) \in \Im(\theta)$

associativity. follows from assoc. in H

$\theta(e_g)$