Proof (c)

Denote $Ker(\theta) = N$

Define $\Phi : \dfrac{G}{N} \to H$ by $\Phi(Ng) = \theta(g)$

It is straightforward to chack that Φ is an isomorphism.

Second Isomorphism Theorem

Let $N \lhd G$. There is a 1-1 correspondance between the subgroups of G which contain N and the subgroups of $\dfrac{G}{N}$, the normal subgroups of G which contain N correspond to normal subgroups of $\dfrac{G}{N}$

Third Isomorphism Theorem

Let $N \lhd G$, HG. Then

a) $NH = \{n \cdot h : n \in N, h \in H\}$ is a subgroup of G and $NH N$

b) $N \cap H \lhd H$

c) $\dfrac{H}{N \cap H} \cong \dfrac{NH}{N}$

§2 Conjugacy classes and the centre.

Definition 2.1

$x, y \in G$ are conjugate if $\exists f \in G : g^{-1}xg = y$

Notation:

$x \sim y$ if they are conjugate.

$x^G = \{g^{-1}xg \mid \forall\ g \in G\}$

e.g. G, abelean, then $x^G = \{x\}$ for any x

Proposition 2.2

Conjugacy is an equivalence relation.

Proof:

(reflexive) $x \sim x : x = 1^{-1}x \cdot 1$

(symmetric) $x \sim y \Rightarrow y \sim x$: if $y = g^{-1}xg$ then $x = gyg^{-1} = (g^{-1})^{-1}y(g^{-1})$

(transitive) if $y = g^{-1}xg$, $z = h^{-1}yh$ then $z = (gh)^{-1}xgh$

Example 2.3

Let $E(2)$ be the group of affine transformations of a plane.

x - translation, $y = g^{-1}xg$ (for some $g \in E(2)$) $\Rightarrow y$ is a translation by a vector of the same length.

x - rotation, $y = g^{-1}xg \Rightarrow y$ is a rotation (possibly around different point) by the angle.

Proposition 2.4

$H \lhd G \Leftrightarrow H$ is a union of some conjugate classes in G

Proof

H is normal means, $g^{-1}Hg = H\ \forall\ g \in G$

i.e. $g^{-1}xg \in H$ for any $x \in H$, $g \in G$, which settles the claim.

Proposition 2.5

Conjugate elements of a group have the same order

Proof

$1 = \left(g^{-1}xg\right)^{n} = g^{-1}xgg^{-1}xg\ldots g^{-1}xg = g^{-1}x^{n}g$

$\Leftrightarrow g = x^{n}g$

$\Leftrightarrow 1 = x^{n}$

Definition 2.6

The centraliser of $x \in G$, $C_G(x) = \{g \in G : gx = xg\}$

The centre of G

$Z(G) = \{x \in G : xg = gx \; \forall \; g \in G\}$

Proposition 2.7

$Z(G) \lhd G$

Proof

$x, y \in Z \Rightarrow xg = gx, yg = gy, \; \forall \; g \Rightarrow xyg = xgy = gxy \; \forall \; g$

$\Rightarrow xy \in Z$

$x \in Z \Rightarrow xg = gx \; \forall \; g \Rightarrow gx^{-1} = x^{-1}g \; \forall \; g \Rightarrow x^{-1} \in Z$

Therefore $Z(G)$ is a subgroup

For every $x \in Z(G)$, $x^{G} = x$, so $Z(G)$ is a union of conjugacy classes, so it is a normal subgroup by proposition 2.4