Centre $Z(G0 = \{x \in G : xg = gx \; \forall \; g \in G\}$

Example

$$G = SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, , b, c, d \in \mathbb{R}, \; ad - bc = 1 \right\}$$

$Z(G)$: Let $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

These are equal if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Z(G)$

$\Rightarrow c = 0, a = d$

Similarly, $g = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ gives, $b = 0, a = d$

So $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $a^2 = 1$, i.e. $\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We showed that $Z(G)$ can contain only $\pm I$ and it is easy to check that $Z(G) = \{\pm I\}$.

$Z(G) \lhd G$ hence $\frac{G}{Z(G)}$ is a grou, which is called $PSL_2(\mathbb{R})$.

Consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \frac{ax+b}{cx+d}$

$+I, -I \rightarrow \frac{1x}{1}$ - the identity transformation.

This defines correspondence (isomorphism) btw $PSL_2(\mathbb{R})$ and the fractional linear transforms.

Theorem 2.8

a) For any $x \in G$, $C_G(x)$ is a subgroup of G.

b) $x^G \overset{1-1}{\leftrightarrow}$ cosets of G modulo $C_G(x)$

c) (the class equation) $\sum_i \frac{1}{|C_G(x_i)|} = 1$

Where x_i runs through the representatives of the conjugacy classes in G.

Proof:

a) If $gx = xg$ and $hx = xh$, then $ghx = gxh = xgh$, so $gh \in C_G(x)$. Similarly, $g^{-1}x = xg^{-1}$. So $C_G(x) \le G$

b) Define $x^G = \{g^{-1}xg\} \leftrightarrow C_G(x)g$ as a formal relation between x^G and $\frac{G}{C_G(x)}$.

Need to show that it is a 1-1, i.e.

$g^{-1}xg = h^{-1}xh \Leftrightarrow C_G(x)g = C_G(x)h$

LHS $\Leftrightarrow xgh^{-1} = gh^{-1}x$, ie $gh^{-1} \in C_G(x)$

RHS $\Leftrightarrow gh^{-1} \in C_G(x)$

So both are equivalent.

c) By (b) and Lagrange's Theorem.

$$\left|x^G\right| = \frac{|G|}{|C_G(x)|}$$

The sum over reps x_i (as above) of $\left|x_i^G\right|$ is $|G|$.

So, $\sum_{x_i} \frac{|G|}{|C_G(x_i)|} = |G|$

$$\sum_{x_i} \frac{1}{|C_G(x_i)|} = 1$$