Let M and N be modules defined over a ring A, a map $f: M \to N$ is a homomorphism if

$$f(x+y) = f(x) + f(y) \ \forall\ x, y \in M$$
$$f(ax) = af(x)\ \forall\ a \in A,\ \forall\ x \in M$$

An invertible homomorphism is called an isomorphism.
For a homomorphism $f: M \to N$, $\Im(f)$, $Ker(f)$ defined as usual

$$\Im(f) = \{f(x) \in N : x \in M\}$$
$$Ker(f) = \{x \in M : f(x) = 0\}$$

These are submodules of N & M respectively.

For a submodule N of M we define the canonical homomorphism to be $\Pi : M \to \frac{M}{N} : x \to x + N$

so $Ker(\Pi) = N$

Theorem 1 (Module Homomorphism Theorem)
Let $f: M \to N$ be homomorphism of A-modules.

Then $\Im(f) \simeq \frac{M}{Ker(f)}$

That is, there exists an isomorphism $\Psi : \Im(f) \to \frac{M}{Ker(f)}$

mapping $y = f(x) \in \Im(f)$ to the coset $\Psi(y) = x + Ker(f)$

Proof - as for abelian groups.

Let $S \subset M$ define the submodule of M generated by S $\langle S \rangle$ to be the set of all linear combinations $a_1x_1 + ... + a_kx_k$ where $x_i \in S$, $a_i \in A$
The operations are induced from M
If M admits a finite generating set S then it is called finitely generated.
If $S = x$, a single element of M, generates M then M is cyclic.

The ideal $Ann(M) = \{a \in M : aM = 0\}$ is called the annihilator of M.
If $Ann(M) \neq \{0\}$ then M is called periodic.

Proposition 1
Every cyclic A-module M is isopmorphic to a module $\frac{A}{I}$ where I is a left ideal of A. If A is commutative then $I = Ann(M)$ and so is defined uniquely.

Proof - easy

A set $\{x_1,..x_n\}$ of elements of M is linearly independent if $a_1x_1 + ... + a_na_n = 0$ implies $a_1 = ... = a_n = 0$.
A linearly independent generating set is called a basis.
A finitely generated module that has a basis is called free.
NB A free cyclic A-module is isomorphic to A.

Just as for finitely generated abelian groups, we can build up a theory of finitely generated modules over principal ideal domains (PIDs)
From now on: Assume A is a PID.

Proposition 2
All bases of a free A-module L contain the same number of elements.

Definition
The number of elements in the basis of a free A-module L is called its rank and denoted $rk(L)$.

Theorem 2
Every submodule N of a free A-module L of rank n is a free A-module rank $m \leq n$. Moreover, there exists a basis $\{e_1,..., e_n\}$ of L and non-zero elements $u_1,..., u_m$ of A so that $\{u_1 e_1,..., u_m e_m\}$ is a basis of N and $u_i | u_{i+1}$ for $i = 1,..., m-1$

Now we study the structure of an arbitrary finitely generated module over a PID A.

- Every non-trivial cyclic A-module is isomorphic to either A itself or to $\frac{A}{(u)}$ for invertible $u \neq 0$.
- If $(u, v) = 1$ then it is east to check $\frac{A}{(u, v)} \cong \frac{A}{(u)} \oplus \frac{A}{(v)}$
- In general if $u = p_1^{k_1} \cdot ... \cdot p_n^{k_n}$ then $\frac{A}{(u)} \cong \frac{A}{\left(p_1^{k_1}\right)} \oplus ... \oplus \frac{A}{\left(p_n^{k_n}\right)}$

Definition
A fin. gen. A-module whose annihilator contains a power of a prime p is called p- primary.

Theorem 3
Every finitely generated A-module M decomposes into a direct sum of primary and free cyclic submodules. Moreover, the collection of annihilators of these submodules is defined uniquely.

Proof
Similar to ableian groups. See 5.22 of Cameron.

We apply Theorem 3 in the case where $A = K[t]$ for a field K. If the resulting vector space is finite dimensional then we only need deal with the finitely generated case.
There are no free summands (- these contain all powerse of t and so are infinite dimensional).

If K is algebraically closed the primary cyclic modules look like $\frac{K[t]}{(t-\lambda)^m}$, which is an m dimensional vector space over K, with basis $\{(t-\lambda)^{m=1},...(t-\lambda), 1\}$
Then the matric of the lienar operator is

$$J(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & \lambda & 1 \\ 0 & 0 & ... & ... & 0 & \lambda \end{bmatrix}$$ - a Jordan block matrix.

Hence
Theorem 4
Every linear operator on a finite dimensional vector space over an algebraically closed field has a Jordan Normal Form in some basis. Moreover this form is unique up to permutation of factors.