11.3 Solution of Laplace equation by separation of variables (cont)

Sep of variable

$$\Rightarrow \Phi = \sum_{n=1}^{\infty} A_n \cdot \sin(n\pi x)\sinh(n\pi y)$$

At $y = 1$, $\Phi(x, 1) = \sum_{n=1}^{\infty} A_n \sinh(n\pi)\sin(n\pi x) = f(x)$ This is a Fourier Series

To find A_n, we need to find the F.S. of f

Using $\int_0^1 \sin(n\pi x)\sin(m\pi x)\, dx = \frac{1}{2}\delta_{nm}$

So $\int_0^1 \sin(m\pi x) f(x)\, dx = \sum_{n=1}^{\infty} A_n \sinh(n\pi) \int_0^1 \sin(m\pi x)\sin(n\pi x)\, dx = A_m \sinh(m\pi)\frac{1}{2}$

$$A_m = \frac{2}{\sinh(m\pi)} \int_0^1 \sin(m\pi x) f(x)\, dx$$

Eg A slight generalisation is to solve for nontrivial B.C. on all 4 sides.

$$\Phi = \Phi_A + \Phi_B + \Phi_C + \Phi_D$$

By uniqueness, this gives the correct solution.
In this way, we can solve the porblem by solving simpler problems whose solution is know already.

11.4 Solving Poisson equation by Green function method

$\nabla^2 \Phi = \rho$ in V

$\Phi|_S = \Psi$

11.4.1 Green Function

Recall δ- function has the property

$$\int_{-\infty}^{b} f(x)\delta(x - x_0)\, dx = f(x_0)$$

We can generalise this to higher dim

$$\int_{\mathbb{R}^d} f(\underline{x})\delta(\underline{x} - \underline{x}_0) d\underline{x} = f(\underline{x}_0)$$

It is east to see that this is just a product of 1d δ.

Definition

A Gree function $G(x)$ satisfies the following defining equation

$$\nabla_x^2 G(x - y) = \delta(x - y),\ x, y \text{ vectors.}$$

An example of Green function will be provided in problem class after Easter

11.4.2 Solving 3d Poisson using Green function

Trick: Use divergence theorem for

$$\begin{aligned}\nabla\cdot(\Phi\nabla G - G\nabla\Phi) &= \nabla\Phi\cdot\nabla G + \Phi\nabla\cdot\nabla G - \nabla G\cdot\nabla\Phi - G\nabla\cdot\nabla\Phi \\ &= \Phi\nabla^2 G - G\nabla^2\Phi \\ &= \Phi\delta(x - y) - G\rho\end{aligned}$$

Div. Theorem

$$\int_V \nabla\cdot(\Phi\nabla G - G\nabla\Phi)\,\mathrm{d}^3x = \int_V \Phi\delta(x-y) - \rho G(x-y)\ \mathrm{d}^3x = \Phi(y) - \int_V G(x-y)\rho(x)\ \mathrm{d}^3x$$

$$\text{LHS} = \int_S \Phi\nabla G - G\nabla\Phi\ \mathrm{d}A = \int_S \Psi\nabla G - G\nabla\Phi\ \mathrm{d}A$$

Now we choose Green function G s.t. $G|_S = 0$ then we obtain

$$\Phi(y) = \int_V G(x-y)\rho(x)\ \mathrm{d}^3x + \int_S \Psi\nabla G\,\mathrm{d}A$$