Theorem 4.6 (Poincaré-Hopf)

Let $S \subset \mathbb{R}^3$ be compact orientable surface possibly with boundary, and $v : S \to \mathbb{R}^3$ a vector field with finitely many singularities $p_1, \ldots, p_k$. Assume that v points outward along the boundary.

Then $\sum_{i=1} I(p_i) = \chi(S)$

Proof

First assume no boundaty.

Then $S = M_g$, $g \geq 0$ by classification thm.

Induction on g : $g = 0$, follows from 4.5

Now assume true for genus $h < g$, need it for genus g.

Embed a circle C into M_g. Make sure that C misses singularities.

Cut the surface open along C.

We get a surface of genus $g - 1$, with 2 disks removed.

Now glue in two disks to make it a suface $g - 1$ without boundary.

Extend the vector field to these disks.

That way er get a vector field $v^{\sim}$ on the surface of genus $g - 1$, so the theorem applies to this case.

Singularities are $p_1, \ldots p_k$ from before, but we also get 2 more singularites in the middle of the disks.

By induction assumption, we get

$$\sum_{i=1} I(p_i) + I(a) + I(b) = \chi(M_{g-1}) = 2 - 2g + 2$$

Take the two disks and put them together to a sphere.

Note the vector fields on the 2 disks match at the boundary, so we get a vector field on the 2-sphere with two singularities a, b

So $I(a) + I(b) = 2$

Hence $\sum_{i=1} I(p_i) = 2 - 2g = \chi(M_g)$

Now assume S has a boundary

Then $S = M_{g,r}$ (surface of genus g with r disks removed).

$r = 1$

Add a disk and extend the vector field as before.

Note that on this disk we get a sink.

Index of this extra singularity c is 1.

$$\sum_{i=1} I(p_i) + I(c) = \chi(M_g) = 2 - 2g$$

So $\sum_{i=1} I(p_i) = 2 - 2g - 1 = \chi(S)$ [QED]

To have a vector field without singularities (and pointing outward on the boundary) we need $\chi(S) = 0$

$2 - 2g - r = 0$

Only cases are T^2 and cyclinder.