Let $\{f_n\}$ be a sequence of functions as a set X. Here is a method (necessary & sufficient) for testing for uniform convergence on X.

Step 1

For each fixed $x \in X$. Does the sequence of numbers $\{f_n(x)\}$ converge? If there is at least on $x \in X$ such that $\{f_n(x)\}$ does not converge, then $\{f_n\}$ doesn't converge pointwise on X and so doesn't converge uniformly on X. However, if $\{f_n(x)\}$ does converge (to $f(x)$) for every $x \in X$, go to

Step 2

For each fixed $n \in \mathbb{N}$: Can we find S_n s.t.

i) $|f_n(x) - f(x)| S_n \ \forall\, x \in X$

ii) $\lim_{n \to \infty} S_n = 0$

If such a sequence $\{n\}$ can be found then $\{f_n\} \xrightarrow{U} f$ on x.

Example

If $f_n(x) = x^n$, does $\{f_n\}$ converge uniformly on $\left[0, \frac{1}{2}\right]$

Step 1

Fix $x \in \left[0, \frac{1}{2}\right]$. Does the sequence x, x^2, x^3 ... of real numbers converge?

Answer: Yes - it converges to 0

So $\{x^n\} \xrightarrow{P} f(x) = 0$ on $\left[0, \frac{1}{2}\right]$

Step 2

Fix $n \in \mathbb{N}$. Then $|f_n(x) - f(x)| = \left|x^n\right| \left(\frac{1}{2}\right)^n \ \forall\, x \in \left[0, \frac{1}{2}\right]$

So take $S_n = \frac{1}{2^n}$. Then $\lim_{n \to \infty} \left(\frac{1}{2^n}\right) = 0$

So the answer is yes. x^n converges uniformely to zero on $\left[0, \frac{1}{2}\right]$

NB When you do the test, make sure you know when x is fixed & when n is fixed

Exercise

Let $f_n(x) = x^n$. Show that $\{f_n\}$ converges uniformely on $(0, p]$ for any $0 < p < 1$.
Show also $\{f_n\}$ does not converge uniformely on $[0, 1)$

Example 2

Let $\{f_n(z)\} = \frac{z}{n}$. Is $\{f_n\}$ uniformely convergent on $\mathbb{C}$?

Step 1

Fix $z \in \mathbb{C}$. Does $\{f_n(z)\}$ converge?

i.e. does $\left\{ \frac{z}{n} \right\}$ converge (as $n \to \infty$)? Ans. Yes for each fixed $z \in \mathbb{C}$. $\lim_{n \to \infty} \left\{ \frac{z}{n} \right\} = 0$. i. e. we have pointwise convergence & pointwise limit is $f(z) = 0$

Step 2

Fiz $n \in \mathbb{N}$. $\left| f_n(z) - f(z) \right| = \left| \frac{z}{n} \right| \; S_n \; \forall z \in \mathbb{C}$ (can't find such an S_n)

Hence Step 2 cannot be carried out - convergence is not uniform on R

Uniform Converge of a series of functions

Definition

A series $\sum_{n=0}^{\infty} f_n$ converges uniformly on a set X if the sequence $S_m = \sum_{n=0}^{m} f_n$ converges uniformly on X.

NB Distinguish between series & sequences

Eg the sequences $\left\{ \frac{1}{n} \right\}$ converges to 0

But the series $\sum_{n=0}^{\infty} \frac{1}{n}$ does not converge.

The test for uniform convergence of series which is analogous to that we've seen for sequences would demand that we know the pointwise limit of the series of functions.

Example

Does $\sum_{n=1}^{\infty} (n+1) z^n$ converge uniformely on $|z| \; \frac{1}{2}$?

Help No idea what pointwise limit might be!

The Weierstrauss M-Test gets over this. You don't need to know the pointwise limit to apply the test.

Weierstrauss M-Test

A series $\sum_{n=0}^{\infty} f_n$ of functions converges uniformly ona set X of real or complex numbers if for each $n \in \mathbb{N}$ we can find a positive real number M_n s.t.

i) $|f_n(z)| \; M_n \; \forall z \in X$

ii) $\sum_{n=0}^{\infty} M_n$ converges

This is good because

a) You don't need to know the pointwise limit

b) It reduces the often hard problem of uniform convergence of a series of functions to the easier problem of convergence of a series of numbers.

Recall there are tests for convergence of a series of real numbers - e.g. ratio test, comparison test.

Power Series

ie. Series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$

Eg. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $\sum_{n=0}^{\infty} z^n$, $\sum_{n=0}^{\infty} \frac{n^2 (z-i)^{2n}}{2^n}$

These are useful next term because you'll see that every holomorphic function can be written (locally) as a power series.

Pointwise convergence properties (Core B1)

Theorem

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then $\exists\ 0\ R\ \infty$ s.t.

i) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges pointwise for $|z-z_0| < R$

ii) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges pointwise for $|z-z_0| > R$

R is the radius of convergence and $|z-z_0| < R$ is the disc of convergence.

You don't know what happens on the circle $|z-z_0| = R$

Uniform Convergence properties

Lemma 1

If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R, then the series converges uniformly on $|z-z_0|\ \rho$ for any $\rho < R$.

Proof

Uses M-Test. First note that $|a_n(z-z_0)^n|\ a_n\rho^n$ on disc of radius ρ. Also, $\sum_{n=0}^{\infty} a_n\rho^n$ converges (why?
)

Lemma 2

If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R, then so has the series of derivatives
$\sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$

Proof

Not too hard - uses comparison test.

Theorem

Assume that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with radius of convergence R. Then the limit is differentiable on $|z-z_0| < R$ and $\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$
ie. you can differentiate a power series term by term.

Proof

Just use Lemma 1, Lemma 2 and Theorem D for series.

Eg. Find radius of convergence R of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

Use ratio test: $\lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\cdot\frac{n!}{z^n}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 0$ (no matter what z is)

So, by ratio test

$\sum \frac{z^n}{n!}$ converges for all z ie. $R = \infty$.

Eg

$$\sum \frac{n^2 (z-i)^{2n}}{2^n}$$

Here $\lim_{n\to\infty} \left| \frac{(n+1)^2 (z-i)^{2n+2}}{2^{n+1}} \cdot \frac{2^n}{n^2 (z-i)^{2n}} \right| = \lim_{n\to\infty} \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 \left|z-i\right|^2$

$CQLT \frac{1}{2} |z-i|^2$ By ratio test, we have convergence $|z-i| < \sqrt{2}$ & divergence $|z-i| > \sqrt{2}$.

Hence radius of convergence is $\sqrt{2}$.