Rouché Theorem

Let f and g be holomorphic inside and on a contour γ and suppose that $|f(z)| > |g(z)|$ on γ. Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside γ.

Proof

Priestley p184.

Example 1

Let $p(z)=z^2-2iz-2$

By the F.T.A, p has 2 zeros in $\mathbb{C}$

Let $p(z)=f(z)+g(z)$ with

$f(z)=z^2$

$g(z)=-2iz-2$

f has 2 zeros inside γ: $|z|=R$

For $z\in\gamma$: $|f(z)|=|z^2|=R^2$

$|g(z)|=|-2iz-2|\le|2iz|+2=2R+2$

We want $|f(z)|>|g(z)|$ for $z\in\gamma$, so it is enough to take R such that

$R^2>2R+2 \Leftrightarrow R^2-2R-2>0$

Take $R=3$.

Then $|f(z)|>|g(z)|$ on $|z|=3$

f & g are holomorphic on $\mathbb{C}$

By Rouché's Theorem $f(z)+g(z)=p(z)$ has the same number of zeros inside γ as $f(z)=z^2$, ie 2.

As $p(z)$ has only 2 zeros, so all its zeros are inside $|z|=3$.

Exc. Confirm this by solving $p(z)=0$.

Example 2

Show that $2+z^2-e^{iz}$ has precisely one zero in the open upper half-plane.

Solution

Let γ be the usual D-shaped contour with radius R.

Set $f(z)=2+z^2$

$g(z)=-e^{iz}$

The zeros of f: $f(z)=0 \Leftrightarrow 2+z^2=- \Leftrightarrow z=\pm i\sqrt{2}$

So f has one zero in the upper half-plane (order 1)

We need to check that $|f(z)|>|g(z)|$ on γ

If $x\in[-R,R]$, $f(x)=2+x^2\ge 2$, $|g(x)|=|-e^{ix}|=1 \}\Rightarrow|f(x)|>|g(x)|$

If z is on the semi-circle $z=R\cdot e^{i\theta}$, $0\le\theta\le\pi$

$|f(z)|=|2+R^2e^{2i\theta}|\ge||R^2e^{2i\theta}|-2|=R^2-2$ (R big enough)

$|g(z)|=|-e^{iR\cdot e^{i\theta}}|=|-e^{iR(\cos(\theta)+i\cdot\sin(\theta))}|=|e^{-R\cdot\sin(\theta)}e^{iR\cdot\cos(\theta)}|=e^{-R\cdot\sin(\theta)}\le$

We need $|f(z)|>|g(z)|$

It is enough to take $R^2-2>1 \Leftrightarrow R^2>3$

We take $R>\sqrt{3}$, the $|f(z)|>|g(z)|$

By Rouché's Theorem, $2+z^2-e^{iz}$ has the same number of zeros as $f(z)=2+z^2$ inside the D shaped contour for any $R>\sqrt{3}$. Therefore $2+z^2-e^{iz}$ jas 1 zero in the open upper-half-plane.

Observation

The zero of $2+z^2$ is not the same as that of $2+z^2-e^{iz}$. Rouché's Theorem tells us that there is on zero in the D-shaped contour but does not give us its location.
We can use numerical methods to get an approximation to the zero.