Extrapolation Methods

Recall Richardson Extrapolation

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{h}{2!}f''(x) + O(h^2)$$

Also

$$f'(x) = \frac{f\left(x+\frac{h}{2}\right) - f(x)}{\frac{h}{2}} + \frac{h}{2\cdot 2!}f''(x) + O(h^2)$$

$$\Rightarrow f'(x) = \frac{2\cdot\left[f\left(x+\frac{h}{2}\right) - f(x)\right]}{\frac{h}{2}} - \frac{f(x+h) - f(x)}{h} + O(h^2)$$

Bernouilli Numbers

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$$

$B_0 = 1$

$B_1 = -\frac{1}{2}$

$B_2 = \frac{1}{6}$

$B_4 = -\frac{1}{30}$

$B_6 = \frac{1}{42}$

$B_8 = -\frac{1}{30}$

Euler-Maclaurin Summation Formula

Let t $[0, N]$ with $N \in \mathbb{N}$, and let $g \in C^{2p+2}[0, N]$. Then:

$$\int_0^N g(t)\,\mathrm{d}t = -\frac{1}{2}[g(o) + g(N)] + \sum_{l=0}^{N} g(l) + \sum_{k=1} \frac{B_{2k}}{(2k)!}[g^{(2k-1)}(0) - g^{(2k-1)}(N)] + R_{p,N}$$

Where

$R_{P,N} = \int_0^N \frac{1}{(2p+2)!} S_{2p+2}(t) g^{(2p+2)}(t)\,\mathrm{d}t$ is asymptotic expansion.

Fix N and let $g(t) = f\left(a + \frac{b-a}{N}t\right)$, $f \in C^{(2p+2)}[a, b]$ given.

then

$$\int_0^N g(t)\,\mathrm{d}t = \frac{1}{h}\int_a^b f(x)\,\mathrm{d}x$$

Now look at the trapezium sum

$$-\frac{1}{2}[g(0)+g(N)]+\sum_{l=0} g(l)=\frac{1}{h}\left[\frac{1}{2}f(a)+f\left(x_1\right)+\ldots+f\left(x_{N-1}\right)+\frac{1}{2}f(b)\right]$$

$$\Rightarrow \int_a^b f(x)\,\mathrm{d}x-h\left\{\frac{1}{2}f(a)+f\left(x_1\right)+\ldots+\frac{1}{2}f(b)\right\}$$

$$=h\sum_{k=1}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(a)-f^{(2k-1)}(b)\right]h^{(2h-1)}+hR_{p,N}$$

$$=\frac{h^2B_2}{2!}\left[f'(a)-f'(b)\right]+\sum_{k=2}\frac{B_{2k}}{(2k)!}[\ldots]h^{2k}+hR_{p,N}(**)$$

Now taking $2N+1$ nodes, ie with subintervals of width $\frac{h}{2}$, we have the formula

$$\int_a^b f(x)\,\mathrm{d}x=\frac{h}{2}\left\{\frac{1}{2}f(a)+f\left(x_{\frac{1}{2}}\right)+f\left(x_1\right)+\ldots+f\left(x_{N-1}\right)+f\left(x_{n-\frac{1}{2}}\right)+\frac{1}{2}f(b)\right\}$$

$$=\frac{h^2}{4}\frac{B_2}{2!}\left[f'(a)-f'(b)\right]+\sum_{k=2}\frac{B_{2k}}{(2k)!}[\ldots]\left(\frac{h}{2}\right)^{2k}+\frac{h}{2}R_{p,2N}(**)$$

4(**)-(*) gives

$$3\int_a^b f(x)\,\mathrm{d}x-2h\left\{\frac{1}{2}f(a)+f\left(x_{\frac{1}{2}}\right)+f\left(x_1\right)+\ldots\right\}+h\left\{\frac{1}{2}f(a)\ldots\right\}$$

$$=\sum_{k=2}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(a)-f^{(2k-1)}(b)\right]\left[\frac{4}{2^{2k}}-1\right]\cdot h^{2k}+2hR_{p,2N}-hR_{p,N}$$

$\frac{1}{3}$ LHS is

$$\int_a^b f(x)\,\mathrm{d}x-h\left\{\frac{1}{6}f(a)+\frac{2}{3}\left(x_{\frac{1}{2}}\right)+\frac{1}{3}\left(x_1\right)+\ldots\right\}=\mathrm{O}\left(h^4\right)+R$$